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ON THE THEORY OF MIXED PROBLEMS FOR THE THREE-DIMENSIONAL WEDGE

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We consider two mixed problems for a wedge, whose edges are infinite in both directions. We investigate the integral equation generated by these problems and we present a method for its solution, different from the known approaches, effective for small apex angles α of the wedge.

The case when the displacement and stress field does not depend on the coordinate along the edge of the wedge corresponds to the plane boundary problem.

A detailed study of the fundamental and mixed problems for a wedge is contained in [1 – 4] and other papers. In the case of a three-dimensional wedge, the static elasticity problem of the second kind has been solved by Ufliand [2] and in the paper by Ulitko (*) the problem of the first kind has been reduced to a single-valued invertible Fredholm equation of the second kind.

1. On the basis of Ia. S. Ufliand's solution [2] we investigate the static mixed problem, antisymmetric with respect to α, with two separation lines of the boundary conditions, parallel to the z-axis which is directed along the edge of the wedge, and situated at distances c and a from it, respectively. In the domain between the mentioned lines, on both sides of the wedge, we assume that we are given the normal pressure depending on z and the displacements parallel to the faces. In the exterior of the indicated domain the faces of the wedge are rigidly fixed.

The second mixed problem is formulated for the anti-plane deformation of the wedge. We assume that one of the faces of the wedge is rigidly fixed, while the other one, in the domain mentioned before, is loaded by a punch whose base performs a harmonic oscillation in the direction of the z-axis, independent of this coordinate. The unique unknown under the punch is the shear stress parallel to the z-axis. The wedge is assumed to be viscoelastic with a Young's modulus constant in time and with a creep kernel $\theta(t-\tau)$ depending on the difference of the arguments. In particular, in the case of the absence of creep, we obtain an elastic wedge.

The described problems can be reduced, with the aid of the Kantorovich-Lebedev transform [2], to the solution of the internal equation of the form

$$\int_1^a k(r, \rho)\, q(\rho)\, d\rho = Af(r) \qquad (1 \leqslant r \leqslant a) \tag{1.1}$$

*) See A. F. Ulitko: The method of proper vector functions in the three-dimensional problems of the theory of elasticity. Author's abstract of doctoral dissertation, Kiev, 1971.

$$k(r,\rho)=\frac{1}{\pi i}\int_{-\infty}^{\infty} I_{-iu}(\varkappa r)\,K_{-iu}(\varkappa\rho)\,K(u)\,u\,du \tag{1.2}$$

In the case of the static problem we use the following notation:

$$f(r)=c_1 I_\mu(\varkappa r)+c_2 K_\mu(\varkappa r)+\varphi(r) \tag{1.3}$$

$$K(u)=u(\operatorname{ch}2\alpha u-\cos 2\alpha)\,[(3-4\nu)\operatorname{sh}2\alpha u+u\sin 2\alpha]^{-1}(u^2+\mu^2)^{-1}$$

$$A^{-1}=4G(1-\nu),\qquad a=d/c$$

$$r^2\varphi''+r\varphi'-(\varkappa^2r^2+\mu^2)\,\varphi=w(r)$$

Here c_1, c_2 are constants subject to determination, r, ρ are dimensionless parameters relative to c, $\mu>0$ is an arbitrary number, $\varphi(r)$ is a particular solution of the differential equation, $q(r)$ and $w(r)$ are the Fourier transforms with respect to the coordinate z of the normal displacement and the stress, respectively, $\varkappa=|\gamma|$, where γ is the parameter of the Fourier transforms with respect to the coordinate z.

In the case of the dynamical problem we have introduced the notation

$$\varkappa^2=-D\sigma^2c^2G^{-1},\qquad K(u)=u^{-1}\operatorname{th}\alpha u,\qquad A=G/c$$

$$\sigma^2=\omega^2\theta^*,\qquad G^\circ=G\theta^*,\qquad \theta^*=1+\int_0^\infty\theta(\tau)\,e^{i\omega\tau}d\tau \tag{1.4}$$

Here ω is the circular frequency of the oscillations of the punch, D is the density of the material of the wedge, $f(r)$ is the amplitude of the displacements of the strip in the contact region, and G°, ν are the shear modulus and Poisson's ratio of the material of the wedge, respectively. The relations between the dimensionless quantities and those with dimensions are the same as in the static problem.

In [5] a method is presented which allows the investigation of the special case when Eq. (1.1) is given on the interval $[0,a]$; but if Eq. (1.1) is given on the interval $[1,a]$, this method is not immediately applicable.

Below we present another method for the solution of this problem, which is based on the reduction to an infinite system of algebraic equations [6].

Without defining concretely the function $K(u)$, we will assume that it is even and real on the real axis, meromorphic in the complex plane, and it does not have real zeros and poles. We assume that it admits a representation of the form

$$K(u)=K_+(u)\,K_-(u),\qquad \lim uK(u)=C\qquad (u\to\infty) \tag{1.5}$$

Here K_+ and K_- are functions regular in the upper and lower half-planes, respectively, decreasing together with the functions $[uK_\pm(u)]^{-1}$ on the system of regular contours.

Let

$$K_+'(-z_l)=C_1 l^{-1/2}\,[1+O(l^{-1}\ln l)] \tag{1.6}$$

$$[K_-^{-1}(\zeta_l)]'=C_2 l^{1/2}\,[1+O(l^{-1}\ln l)]\qquad (l\to\infty)$$

Here z_l and ζ_l are the zeros and the poles of the function $K(u)$, respectively, situated in the upper half-plane and assumed to be simple with a finite distribution density β. In order to simplify Eq. (1.1), we represent its right-hand side by a Kantorovich-Lebedev integral and therefore we restrict ourselves to the case

$$f(r)=I_\eta(\varkappa r)\,I_\eta^{-1}(\varkappa a)$$

For the indicated right-hand side we will seek the solution of Eq. (1.1) in the form of the series

$$q(\rho) = A\{x_0 I_\eta(\varkappa\rho) I_\eta^{-1}(\varkappa a) + \sum_{k=1}^{\infty} [x_k I_{-iz_k}(\varkappa\rho) I_{-iz_k}^{-1}(\varkappa a) + y_k K_{-iz_k}(\varkappa\rho) K_{-iz_k}^{-1}(\varkappa)]\} \rho^{-1} \tag{1.7}$$

where x_k, y_k are constants subject to determination. We insert (1.7) into the left-han side of Eq. (1.1) and we perform the integration representing the kernel $k(r, \rho)$ in the form

$$k(r, \rho) = \sum_{k=1}^{\infty} s_k \begin{pmatrix} K_{-i\zeta_k}(\varkappa r) I_{-i\zeta_k}(\varkappa\rho), & r > \rho \\ I_{-i\zeta_k}(\varkappa r) K_{-i\zeta_k}(\varkappa\rho), & r < \rho \end{pmatrix} \tag{1.8}$$

This representation is obtained as a result of the evaluation of the integral (1.2) with the theory of residues. The latter is possible due to the estimates regarding the behavi of the functions $K_p(z)$ and $I_p(z)$ in the complex plane p, obtained on the basis of th results given in [7] and having the form [8] ($|p| \to \infty$)

$$\begin{gathered} I_p(z) = 1/\pi\sqrt{2}(z^2 + p^2)^{-1/4} e^{\omega}(1 + O(p^{-1})) \\ K_p(z) = 1/\sqrt{2}(z^2 + p^2)^{-1/4} e^{-\omega}(1 + O(p^{-1})) \\ (\omega = \sqrt{z^2 + p^2} - p \operatorname{arsh} pz^{-1}) \end{gathered} \tag{1.9}$$

As a result of the integration we obtain Dirichlet type series with respect to the functic $I_p(z)$ and $K_p(z)$. Since these series must give the right-hand side of Eq. (1.1), we ar rive to an infinite system for the determination of the constants x_l and y_l

$$\begin{gathered} A_{11}X + A_{12}Y = B_1, \qquad X = \{x_l\} \\ A_{21}X + A_{22}Y = B_2, \qquad Y = \{y_l\} \\ A_{11} = \{a_{rl}(1,1)\} = iW[I_{-iz_l}(\lambda) K_{-i\zeta_r}(\lambda)][(\zeta_r^2 - z_l^2) I_{-iz_l}(\lambda) K_{-i\zeta_r}(\lambda)]^{-1} \\ A_{12} = \{a_{rl}(1,2)\} = iW[K_{-iz_l}(\lambda) K_{-i\zeta_r}(\lambda)][(\zeta_r^2 - z_l^2) K_{-iz_l}(\varkappa) K_{-i\zeta_r}(\lambda)]^{-1} \\ A_{21} = \{a_{rl}(2,1)\} = -iW[I_{-iz_l}(\varkappa) I_{-i\zeta_r}(\varkappa)][(\zeta_r^2 - z_l^2) I_{-iz_l}(\lambda) I_{-i\zeta_r}(\varkappa)]^{-1} \\ A_{22} = \{a_{rl}(2,2)\} = -iW[K_{-iz_l}(\varkappa) I_{-i\zeta_r}(\varkappa)][(\zeta_r^2 - z_l^2) K_{-iz_l}(\varkappa) I_{-i\zeta_r}(\varkappa)]^{-1} \\ B_1 = \{b_r(1)\} = ix_0 W[K_{-i\zeta_r}(\lambda) I_\eta(\lambda)][(\zeta_r^2 + \eta^2) K_{-i\zeta_r}(\lambda) I_\lambda(\lambda)]^{-1} \\ B_2 = \{b_r(2)\} = -ix_0 W[I_{-i\zeta_r}(\varkappa) I_\eta(\varkappa)][(\zeta_r^2 + \eta^2) I_{-i\zeta_r}(\varkappa) I_\eta(\lambda)]^{-1} \\ W[x, y] = x'y - y'x, \quad x_0 = K^{-1}(i\eta), \quad \lambda = \varkappa a \end{gathered} \tag{1.10}$$

The equalities (1.10) are sufficient conditions for the solvability of Eq. (1.1) in the class of solutions represented in the form (1.7). These conditions turn out to be also necessary, provided the system $I_{\lambda_k}(z)$, $K_{\lambda_k}(z)$ ($k = 1, 2, \ldots$) possesses the minimality property (strong linear independence) on the interval of the real axis not containing the origin. In the case under consideration, the minimality property for the given system can be proved by constructing the transformation operators [9], and applying them to the minimal system of exponential functions ([10], p. 133).

It is easy to verify that for $|\zeta_r| \to \infty$, $|z_l| \to \infty$ the elements of the matrices A_{kk} tend to the matrix A with the elements $(\zeta_r - z_l)^{-1}$, while the elements of the matrice A_{kj}, $k \neq j$ vanish. The infinite system with the matrix A has been analyzed in [11] We will use the results of that paper to the investigation of the system (1.10).

2. Making use of the inverse matrix A^{-1} [11], the system (1.10) can be reduced to the normal form

$$X = A^{-1}(A - A_{11})X - A^{-1}A_{12}Y + A^{-1}B_1$$
$$Y = A^{-1}(A - A_{22})Y - A^{-1}A_{21}X + A^{-1}B_2 \qquad (2.1)$$

With the aid of the estimates (1.6) and (1.9) we can establish that the matrices in the right-hand side of the system (2.1) generate completely continuous operators in the space of the sequences $s(\sigma)$, $(0 < \sigma < 1/2)$. Here $X \in s(\sigma)$, under the conditions

$$\| X \|_{s(\sigma)} = \sup_k | x_k k^\sigma | < \infty, \qquad \lim | x_k k^\sigma | = 0 \qquad (k \to \infty)$$

In a number of cases one succeeds to prove the unique solvability of the system (2.1) in $s(\sigma)$. The latter takes place, for example, for $\varkappa > 0$. In this case, the operator in the left-hand side of (1.1) is positive definite in some Hilbert space. In the general case the system (2.1) is quasi-regular and it can be studied by the known methods [12].

For small apex angles of the wedge, the operators in the right-hand side of the system (2.1) are small and they can be made to be contractive; the system can be solved by the method of successive approximations.

3. We investigate the zero-th approximation of the solution of the system (2.1). Taking into account what has been said above, we note that such a solution is effective for small apex angles of the wedge. Computing the elements of the matrix $A^{-1}B_k$ and inserting their values into the relation (1.7), we obtain the approximate solution of Eq. (1.1), where the coefficients x_l and y_l have the form

$$x_l = x(z_l) = \frac{1}{K_+'(-z_l)\,2\eta}\left[\frac{V(\lambda) - \eta}{(z_l + i\eta)K_-(i\eta)} - \frac{V(\lambda) + \eta}{(z_l - i\eta)K_+(i\eta)}\right] \qquad (3.1)$$
$$y_l = y(z_l) = \frac{1}{K_+'(-z_l)\,2\eta}\left[\frac{V(\varkappa) - \eta}{(z_l - i\eta)K_+(i\eta)} - \frac{V(\varkappa) + \eta}{(z_l + i\eta)K_-(i\eta)}\right]\frac{I_\eta(\varkappa)}{I_\eta(\lambda)}$$
$$V(x) = I_\eta'(x)\, I_\eta^{-1}(x)$$

For the computation of the function $q(\rho)$ far from the points $\rho = 1$ and $\rho = a$, in the relation (1.7) we can restrict ourselves to a finite number of terms. The series converges as a geometric progression. For the investigation of the function $q(\rho)$ near the indicated points, the relation (1.7) can be summed into an integral and is transformed into the formula of operational calculus

$$q(\rho) = A\{x_0 I_\eta(\varkappa\rho) I_\eta^{-1}(\varkappa a) - (2\pi i)^{-1}\int_\Gamma [x(-t) I_{it}(\varkappa\rho) I_{it}^{-1}(\varkappa a) +$$
$$y(-t) K_{it}(\varkappa\rho) K_{it}^{-1}(\varkappa)]\, K_+'(t) K_+^{-1}(t)\, dt\}\,\rho^{-1} \qquad (3.2)$$

The contour Γ lies in the lower half-plane, enveloping from above the origin and the poles of the function $K_+(z)$ and from below the points $t = \pm i\eta$ Replacing the ratio of the Bessel functions by their asymptotic expressions (1.9), we obtain a first approximation expression for $q(\rho)$ at $\rho \to 1$ and $\rho \to a$, respectively

$$q(\rho) \sim v(1 - \rho^{-\beta})^{-1/2}(1 + O(\ln \rho))$$
$$q(\rho) \sim w[1 - (a/\rho)^{-\beta}]^{-1/2}(1 + O(\ln a/\rho)) \qquad (3.3)$$
$$v = \frac{1}{2\eta\sqrt{\pi C}}\left[\frac{V(\lambda) - \eta}{K_-(i\eta)} - \frac{\eta + V(\lambda)}{K_+(i\eta)}\right]$$
$$w = \frac{1}{2\eta\sqrt{\pi C}}\left[\frac{V(\varkappa) - \eta}{K_+(i\eta)} - \frac{V(\varkappa) + \eta}{K_-(i\eta)}\right]\frac{I_\eta(\varkappa)}{I_\eta(\lambda)}$$

in the case of the static problem, the solution $q(\rho)$ of the integral equation (1.1) depends on two arbitrary constants c_1 and c_2, which can be found from the condition of the boundedness of the displacements at the points $\rho = 1$ and $\rho = a$.

The method of this paper can be generalized to the case of several regions of contact. In this case, at each of the regions where the integral equation (1.2) is given, its solution must be sought in the form of the series (1.7) with its own coefficients x_k and y_k on each of the regions.

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